Matrices (Hints and numerical answers)

1. $\mathbf{A}+\mathbf{B}=\begin{pmatrix}-1 & 2 & 2\\ 9 & 7 & 8\\ 2 & -1 & 3\end{pmatrix}$ $\quad \mathbf{A}-\mathbf{B}=\begin{pmatrix}5 & 0 & -2\\ 3 & -3 & -6\\ 0 & -1 & -1\end{pmatrix}$ $\quad \mathbf{B}-\mathbf{A}=\begin{pmatrix}-5 & 0 & 2\\ -3 & 3 & 6\\ 0 & 1 & 1\end{pmatrix}$

$\mathbf{A}\times\mathbf{B}=\begin{pmatrix}2 & 2 & 5\\ 28 & 6 & 19\\ -2 & 0 & 2\end{pmatrix}$ $\quad \mathbf{B}\times\mathbf{A}=\begin{pmatrix}-3 & 2 & 1\\ 19 & 16 & 27\\ 1 & -4 & -3\end{pmatrix}$

2. **(i)** $\begin{pmatrix}2 & 1\\ 8 & -1\end{pmatrix}$ **(ii)** $\begin{pmatrix}2\\ -6\end{pmatrix}$ **(iii)** $(8 \quad 2)$ **(iv)** $\begin{pmatrix}4 & 2\\ 3 & 1\end{pmatrix}$

3. $\phi(A) = -2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}-5\begin{pmatrix}1 & 2\\ 3 & 1\end{pmatrix}+3\begin{pmatrix}1 & 2\\ 3 & 1\end{pmatrix}^2 = -2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}-5\begin{pmatrix}1 & 2\\ 3 & 1\end{pmatrix}+3\begin{pmatrix}7 & 4\\ 6 & 7\end{pmatrix}=\begin{pmatrix}14 & 2\\ 3 & 14\end{pmatrix}$

4. Geometrical interpretation : rotation of a vector by an angle ϕ clockwisely and then an angle θ clockwisely is the same as rotation of an angle $(\theta + \phi)$ clockwisely. (or rotation of system of axis anti-closkwisely)

5. Let P(n) be the proposition : $\mathbf{A}^n = \begin{pmatrix}1 & n & \frac{1}{2}n(n-1)\\ 0 & 1 & n\\ 0 & 0 & 1\end{pmatrix}$

For P(1), $\mathbf{A}^1 = \begin{pmatrix}1 & 1 & \frac{1}{2}1(1-1)\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}$ $\quad\therefore$ P(1) is true.

Assume P(k) is true for some $k \in \mathbf{N}$, $\mathbf{A}^k = \begin{pmatrix}1 & k & \frac{1}{2}k(k-1)\\ 0 & 1 & k\\ 0 & 0 & 1\end{pmatrix}$

For P(k +1) ,

$$\mathbf{A}^{k+1} = \mathbf{A}^k\mathbf{A} = \begin{pmatrix}1 & k & \frac{1}{2}k(k-1)\\ 0 & 1 & k\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & k+1 & k+\frac{1}{2}k(k-1)\\ 0 & 1 & k+1\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & k & \frac{1}{2}(k+1)(k)\\ 0 & 1 & k\\ 0 & 0 & 1\end{pmatrix}$$

$\therefore$ P(k+1) is true.

By the Principle of Mathematical Induction , P(n) is true $\forall\ n \in \mathbf{N}$.

8. The given simultaneous equations is an inconsistent system.

9. Put $\mathbf{X} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$, $\quad \mathbf{X} = \begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}3 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix}$

$\mathbf{X} = \begin{pmatrix}2+p & q(1\pm p)\\ (1\mp p)/q & 2-p\end{pmatrix}$, where $p, q \in \mathbf{R}$, $q \neq 0$.

10. $A = \begin{pmatrix} 1 & 1/2 & 0 \\ 0 & 1 & 0 \\ 1 & -1/4 & 1 \end{pmatrix}$ **11.** $A^{-1} = \begin{pmatrix} -7 & -4 & 9 \\ 6 & 3 & -7 \\ 3 & 2 & -4 \end{pmatrix}$

12. $(\mathbf{A} + \mathbf{A})^{-1} = \begin{pmatrix} -3.5 & 2 & 4.5 \\ 3 & 1.5 & -3.5 \\ 1.5 & 1 & 2 \end{pmatrix}$ $\mathbf{A}^{-1} + \mathbf{A}^{-1} = \begin{pmatrix} -14 & -8 & 18 \\ 12 & 6 & -14 \\ 6 & 4 & -8 \end{pmatrix}$

13. Consider $(\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB}) = (\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \mathbf{I}$, **SPS'** $= \begin{pmatrix} -2 & 0 \\ 0 & 4 \end{pmatrix}$

14. Write the infinite series for $\sin\theta$ and $\cos\theta$:

$\sin\theta = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots.$, $\cos\theta = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots.$

15. **(i)** $(\mathbf{I} - \mathbf{A})^2 = \mathbf{I} - 2\mathbf{A} + \mathbf{A}^2 = \mathbf{I} - 2\mathbf{A} + \mathbf{A} = \mathbf{I}$ **.**

(ii) **A** is non-singular $\Rightarrow$ $\mathbf{A}^{-1}$ exists

$\mathbf{I} - \mathbf{A} = \mathbf{A}\mathbf{A}^{-1} - \mathbf{A} = \mathbf{A}^2\mathbf{A}^{-1} - \mathbf{A}$ (since **A** is idempotent)

$= \mathbf{A}(\mathbf{A}\mathbf{A}^{-1}) - \mathbf{A} = \mathbf{AI} - \mathbf{A} = \mathbf{A} - \mathbf{A} = \mathbf{0}$.

16. $\mathbf{X}^2 = \begin{pmatrix} a^2 + bc & ab + bd \\ ca + dc & cd + d^2 \end{pmatrix}$ We get the equations :

$a^2 + bc + pa + q = 0$ (1)

$(a + d + p)b = 0$ (2)

$c(a + d + p) = 0$ (3)

$d^2 + bc + pd + q = 0$ (4)

(i) If $a + d + p = 0$, then $\mathrm{Tr}(\mathbf{A}) = a + d = -p$ and from (1) + (4), $|\mathbf{X}| = q$.

(ii) If $a + d + p \neq 0$, then $b = c = 0$. From (1) and (4),

$a^2 + pa + q = 0$ and $d^2 + pd + q = 0$. Result follows.

17. $x = \pm 3$, $\mathbf{B} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$, $\mathbf{B}^{-1}\mathbf{AB} = \begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix}$, $\mathbf{A}^n = \frac{1}{3}\begin{pmatrix} 3^n + 2(-3)^n & 3^n - (-3)^n \\ 2[3^n - (-3)^n] & 2(3^n) + (-3)^n \end{pmatrix}$.

19. $\lambda = 0, -1, 2$. $\vec{x} = (t \quad 0 \quad -t)$, where $t \in \mathbf{R}$.

20. $\alpha = 2r$, $\delta = r^2 - s^2$, $\mathbf{A}^2 = \alpha\mathbf{A} - \delta\mathbf{I} = 2r\mathbf{A} - (r^2 - s^2)\,I$.

21. $(\mathbf{I} - \mathbf{A})(\mathbf{I} + \mathbf{A}) = \mathbf{I} - \mathbf{A}^2 = (\mathbf{I} + \mathbf{A})(\mathbf{I} - \mathbf{A})$ **.**

24. $\mathbf{P}^{-1} = \begin{pmatrix} 1 & -\sqrt{3} & 0 \\ \sqrt{3} & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

25. $\mathbf{A}^{-1} = \frac{1}{60}\begin{pmatrix} -4 & 1 & 13 \\ 16 & -19 & -7 \\ 4 & 14 & 2 \end{pmatrix}$

$x_1 = (1/60)[-4h_1 + h_2 + 13h_3]$

$x_2 = (1/60)[16h_1 - 19h_2 - 7h_3]$

$x_3 = (1/60)[2h_1 + 7h_2 + h_3]$

26. **(i)** $x_1 = 5/4$, $x_2 = -3/4$, $x_3 = 7/2$

(ii) $x_1 = a$, $x_2 = 14 - 3(a + b)$, $x_3 = 2(a + b) - 10$, $x_4 = b$, $a, b \in \mathbf{R}$.

(iii) inconsistent system .

27. **(i)** $x_1 = 13/4$, $x_2 = 1/4$

(ii) $x_1 = 2/3$, $x_2 = -1/3$

(iii) $x_1 = 9/2$, $x_2 = -1/2$, $x_3 = -7/2$

(iv) $x_1 = -1/2$, $x_2 = -1/2$, $x_3 = 3/2$

28. $x_1 = \frac{(\lambda_3 - \lambda_1)(a + \lambda_1)(b + \lambda_1)}{(\lambda_2 - \lambda_1)(a + \lambda_3)(b + \lambda_3)}\mu \quad x_2 = \frac{(\lambda_1 - \lambda_3)(a + \lambda_2)(b + \lambda_2)}{(\lambda_2 - \lambda_1)(a + \lambda_3)(b + \lambda_3)}\mu \quad x_3 = \mu$

29. One solution : $a^2 \neq b^2$ and $b^2 \neq c^2$ and $c^2 \neq a^2$.

Infinite many solutions : $(a = b \neq -c)$ or $(b = c \neq -a)$ or $(c = a \neq -b)$ or $(a = b = c)$

No solution : other cases.

30. Non-zero solution if $(a = b = c)$ or $(a = b = 1$ and $c \neq 1)$ or $(b = c = 1$ and $a \neq 1)$ or $(c = a = 1$ and $b \neq 1)$ or $(a = b \neq 1$ and $c = 1)$ or $(b = c \neq 1$ and $a = 1)$ or $(c = a \neq 1$ and $b = 1)$.

31. **(1)** $b = 0$: no solution

(2) $a = 1$ and $b = 1$: infinite number of solutions

$a = 1$ and $b \neq 1$: no solution

(3) $a = -2$ and $b = -2$: infinite number of solutions

$a = -2$ and $b \neq -2$: no solution

(4) $b \neq 0$, $a \neq 1$, $a \neq 2$: unique solution

32. **(i)** $x = 3$, $y = 1$, $z = 2$.

(ii) inconsistent system.